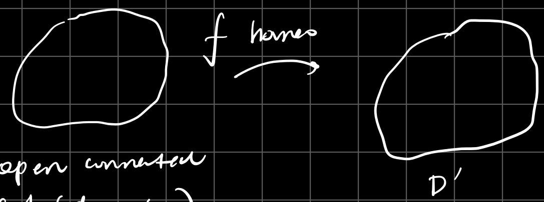
Conformal Maps

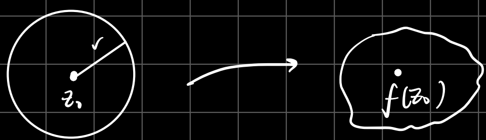


open connected set (domain)
$D \subseteq \mathbb{C}$

Defn. We say that f is CONFORMAL if $\forall z_0 \in D$,

$$\lim_{z \to z_0} \frac{f(z)-f(z_0)}{z-z_0} \quad \text{i.e. } f'(z_0)$$

exists and is nonzero.



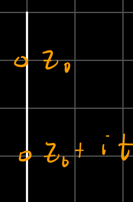
$|f(z)-f(z_0)| \approx |f'(z_0)|$
if $|z-z_0| = r$ - small

- takes circles to circles instead of circles to ellipses

- f sends infinitesimal circles to infinitesimal circles.

- $z = x+iy$
  $f(z) = u(x,y) + iv(x,y)$

$$f'(z_0) = \frac{\partial u}{\partial x}(x_0,y_0) + i\frac{\partial v}{\partial x}(x_0,y_0)$$



$$f'(z_0) = \lim_{t\to 0} \frac{u(x_0,y_0+t)-u(x_0,y_0)}{it} + i\lim_{t\to 0}\frac{v(x_0,y_0+t)-v(x_0,y_0)}{it}$$

$$= -i\frac{\partial u}{\partial y}(x_0,y_0) + \frac{\partial v}{\partial y}(x_0,y_0)$$

CAUCHY-RIEMANN EQUATIONS
$$\begin{cases} \dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial y} \\[2mm] \dfrac{\partial v}{\partial x} = -\dfrac{\partial u}{\partial y} \end{cases}$$

- $$\begin{cases} z = x+iy \\ \bar{z} = x-iy \end{cases} \longrightarrow \begin{cases} x = \frac{1}{2}(z+\bar{z}) \\ y = \frac{1}{2i}(z-\bar{z}) \end{cases}$$

- $\dfrac{\partial}{\partial z} = \dfrac{1}{2}\left(\dfrac{\partial}{\partial x} - i\dfrac{\partial}{\partial y}\right)$

  $\dfrac{\partial}{\partial \bar z} = \dfrac{1}{2}\left(\dfrac{\partial}{\partial x} + i\dfrac{\partial}{\partial y}\right)$

$f = u + iv$

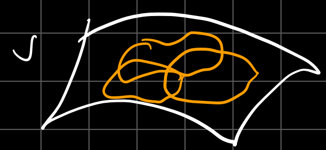
$$\frac{\partial f}{\partial \bar z} = \frac{\partial u}{\partial \bar z} + i\frac{\partial v}{\partial z}$$
$$= \frac{1}{2}\left(\frac{\partial u}{\partial x} + i\frac{\partial u}{\partial y}\right) + i\,\frac{1}{2}\left(\frac{\partial v}{\partial x} + i\frac{\partial v}{\partial y}\right)$$
$$= \frac{1}{2}\left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right) + \frac{i}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)$$
$$= 0 \quad \text{(by CR)}$$

Or, equivalently, $\dfrac{\partial f}{\partial \bar z} = 0$. $\left(\dfrac{\partial f}{\partial z} = f'\right)$.

- Conformal maps preserve angles.
- Nonex) constant maps

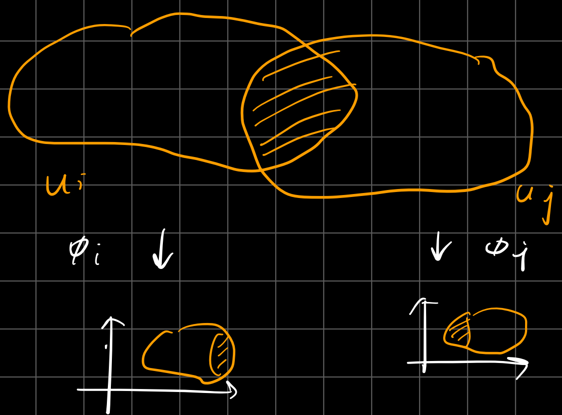
Surfaces

- Connected, orientable, 2-manifold



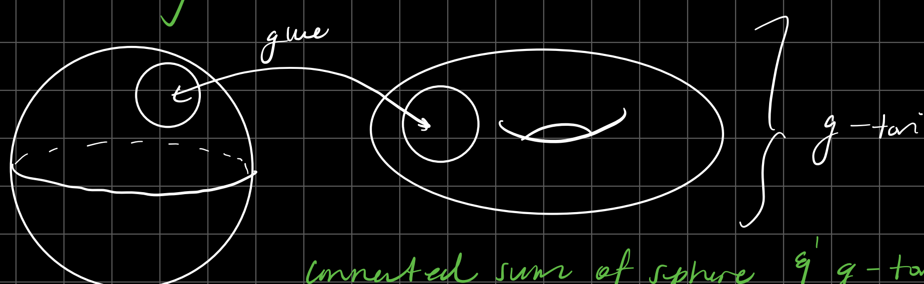
$\underset{\text{open}}{U_i} \subset S$, $\bigcup_i U_i = S$

$\phi_i : U_i \to \mathbb{R}^2$ inj. cont.



$\phi_j \circ \phi_i^{-1}$ is C^∞.

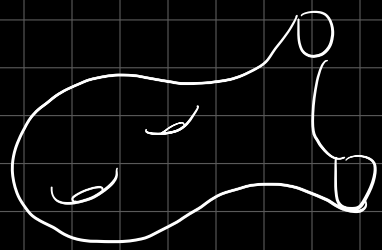
- Defn. A CLOSED SURFACE is a surface S that is compact and has no boundary.



connected sum of sphere & g-tori.

A COMPACT SURFACE is obtained from a closed surface by removing a finite number b of open topological disks.

→ Allowed to have boundary components.



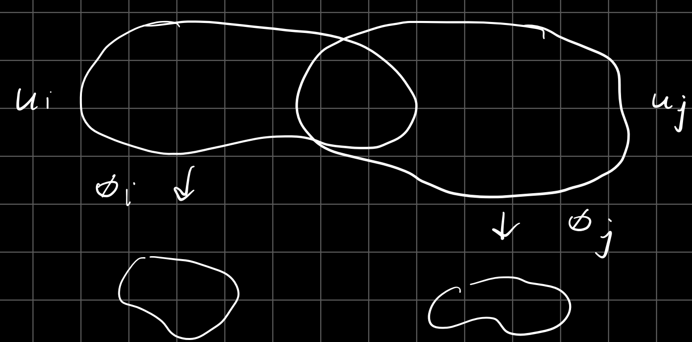
genus ⟷ handles

A SURFACE WITH PUNCTURES is a compact surface with finitely-many points (from interior) removed or marked
- removed → not cpt anymore
- marked → remains cpt

Riemann Surfaces

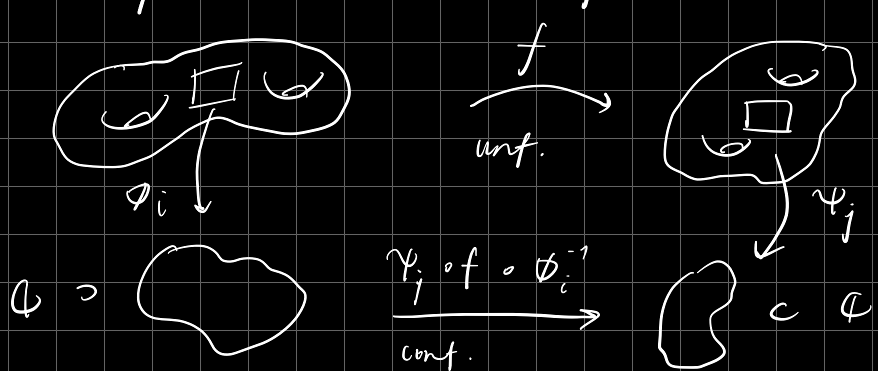
• Defn. A surface S is a RIEMANN SURFACE with the atlas $\mathcal{A} = \{U_i, \phi_i\}$ such that each transition map is conformal.

→ Stronger than C^∞

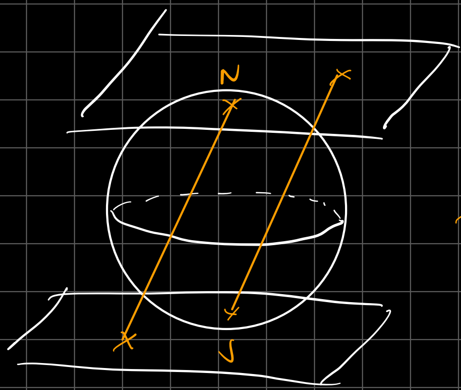


$\phi_j \circ \phi_i^{-1}$ conformal.

• Conformal maps btwn Riemann surfaces



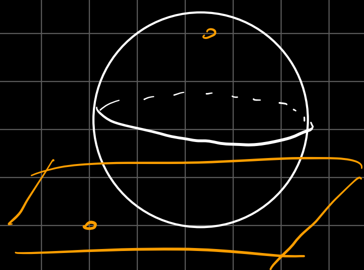
→ Doesn't depend on charts b/c comp of conf. maps is conf.



Stereographic Projection
Riemann.

3 Geometries $\rightsquigarrow$ invariant under isometries.

① Spherical $\rightsquigarrow$ have a notion of distance btwn pts & angles
$\rightsquigarrow$ use conformal metric (type of Riemannian metric)

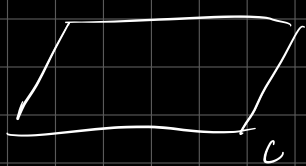


$$\frac{4(dx^2+dy^2)}{(1+x^2+y^2)} = ds^2 \quad \Big| \quad \left(\frac{2|dz|}{1+|z|^2}\right)^2$$

constant (Gaussian) curvature.

★ Universal cover of S^2 is itself

② Planar



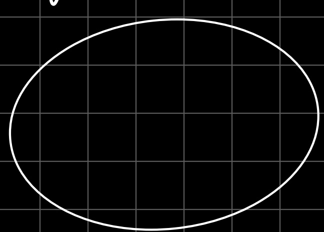
$$ds^2 = dx^2 + dy^2$$

constant curvature

★ Universal cover of T^2 is $\mathbb{R}^2$.

diff. 2-forms
action on tangent spaces.

③ Hyperbolic



$\mathbb{D}$ - unit disk in $\mathbb{C}$

$$ds^2 = \frac{4(dx^2+dy^2)}{(1-x^2-y^2)^2} = \left(\frac{2|dz|}{1-|z|^2}\right)$$

$$a\,dx^2 + 2b\,dx\,dy + c\,dy^2 \longleftrightarrow (u_1 \ u_2)\begin{bmatrix} a & b \\ b & c \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \rightarrow \text{scalar}$$

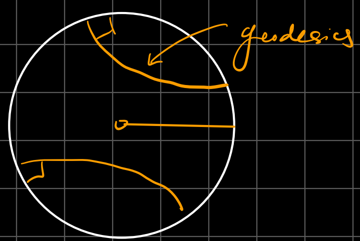
$\lambda(z)^2 |dz|^2$ CONFORMAL METRIC.

Exercise. Hyperbolic metrics are invariant under isometries.
(orientation-preserving Möbius maps preserving the unit disk)

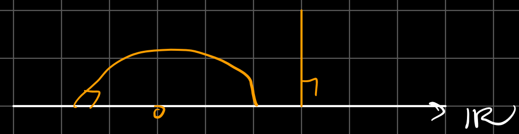
MÖBIUS MAP, or FRACTIONAL LINEAR MAP $\gamma(z) = \dfrac{az+b}{cz+d}$.

Hyperbolic Space Models

→ Poincaré disk



→ upper half plane



$$ds^2 = \left(\frac{|dz|}{y^2} \right)^2$$

Möbius map
Isometry

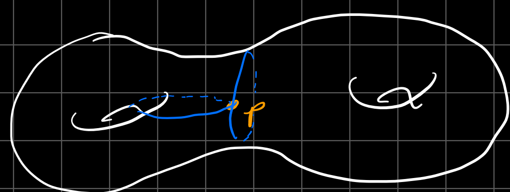
Thm. (Riemann Uniformization) Let S be a Riemann surface such that $\partial S \neq \emptyset$.

$\tilde{S} / \pi_1(S)$ ← universal cover, fund group

$\downarrow$

S

Defn. FUNDAMENTAL GROUP



$\pi_1(S)\{[\gamma]\}$

$\gamma_1 \sim \gamma_2$ if γ_1 is homotopic to γ_2.

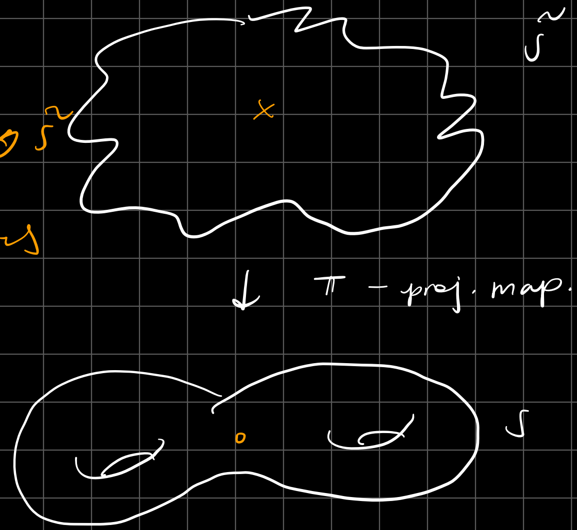
p = base point.

$f, g : X \to Y$ btwn top. spaces

$f \sim g$ HOMOTOPIC if $\exists F : X \times [0,1]$ → Y cont.
$F(x, 0) = f(x)$
$F(x, 1) = g(x)$.

- $\pi_1(\tilde{S}) = \{e\}$

$\pi_1(S) \circlearrowright \tilde{S}$
(lift)
Action by isometries



<u>Thm.</u> If $\tilde{S}$ is a simply-connected ($\pi_1(\tilde{S}) = \{e\}$) Riemann surface, then it is conformal to either the Riemann sphere, complex plane or unit disk (mutually exclusive).
open   in $\mathbb{C}$

<u>Cor.</u> Any S-Riemann surface ($\partial S = \emptyset$) can be endowed with constant curvature conformal metric.

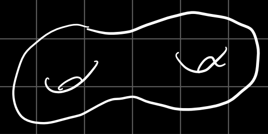
Curvature $= \pm 1, 0$.

Next time... Isometries on $\mathbb{H}^n$ & Teichmüller space (compute for T^2)

Lecture 2 – Thursday, February 6, 2025

- $\tilde{X}$ universal cover

$\tilde{X} \xrightarrow{p} X$



X Riemann surface (closed)

$\pi_1(X)$ fundamental group of X

conformal structure (in terms of charts)

$\tilde{X} \underset{\text{conf equiv.}}{\approx} \bar{\mathbb{C}}$ (Riemannian Sphere), $\mathbb{C}$ or $\mathbb{D}$

$\mathbb{D}$: $\left(\frac{2|dz|}{1-|z|^2}\right)^2$

- $\pi_1(X) \circlearrowright \tilde{X}$ (by Möbius trans.) freely (fixed point free), properly discontinuously (discretely),

  & by isometries (by conformal maps).

- $\begin{cases} \tilde{X} = \bar{\mathbb{C}} & \text{only when } X = \bar{\mathbb{C}} \\ \tilde{X} = \mathbb{C} & \text{only when } X = \mathbb{C},\ \mathbb{C}\setminus\{pt\} \\ \tilde{X} = \mathbb{T}^2 & \leftarrow \text{constant curvatured surface.} \\ \tilde{X} = \mathbb{D} & \text{all other cases.} \end{cases}$

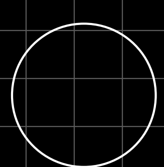
- Euler Characteristic $\chi(X) = 2-2g-(b+n) < 0$

  g = genus
  b = boundary components
  n = # of punctures or marked pts.

Cor. (to the Riemannian Uniformization Thm)
X = arbitrary Riemann surface
It has a constant curvature (-1, 0, or 1) metric.

Isometries on the Hyperbolic Plane
($\mathbb{H}$, $\mathbb{D}$)

- $\mathrm{Isom}^+(\mathbb{H}) = ?$
  orientation-preserving isometries of $\mathbb{H}$.



Conformal self maps of $\mathbb{D}$?
→ use Schwarz lemma. these are all such maps
→ Möbius transformation $(az+b)/(cz+d)$

$\Rightarrow \pi_1(X)$ acts on $\mathbb{D}$ (hyp. surf) by isometries.

★ In the hyperbolic space, conformal maps and isometries are the same.

- $\mathrm{Isom}^+(\mathbb{H}) = \left\{ \dfrac{az+b}{cz+d} \;\middle|\; a,b,c,d \in \mathbb{R} \right\}$
  By scaling, we may assume $ad-bc=1$.

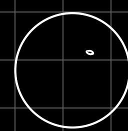
$$\frac{az+b}{cz+d} \longleftrightarrow \begin{bmatrix} a & b \\ c & d \end{bmatrix} = A$$

$\det A = 1.$

$\mathrm{Isom}^+(\mathbb{H}) = PSL(2,\mathbb{R})$
proj. special linear group.
$SL(2,\mathbb{R})/\pm I$

Classification of Isometries of $\mathbb{H}$ (or $\mathbb{D}$).

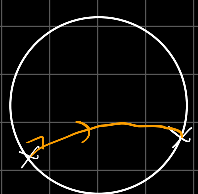
① Elliptic : 1 fixed point in $\mathbb{D}$, conjugate to rotation
($|\operatorname{trace} A| < 2$)



② Parabolic : 1 fixed point on $\partial\mathbb{D}$ ($|\operatorname{trace} A| = \pm 2$)

e.g. $z \mapsto z+1$
∞ fixed pt.

③ Hyperbolic / loxodromic : 2 distinct fixed pts on $\partial\mathbb{D}$



unique geodesic, fixed pts $\Rightarrow$ fixed geodesic
(translates points along fixed geodesic)

($|\operatorname{trace} A| > 2$).

Exercise. Check the trace characteristics.

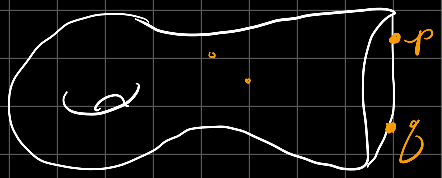
Teichmüller Space.

- S = topological surface (g, b, n)
  $\chi(S) < 0$

$S \xrightarrow{f} X \;\Big|\; (X, f)$

compact Riemann surface with marked points

∂X is totally geodesic | conformal structure.



$\forall p, q \in \partial X$ connected component

$\exists \gamma$ - geodesic boundary joining p and q in the same component.

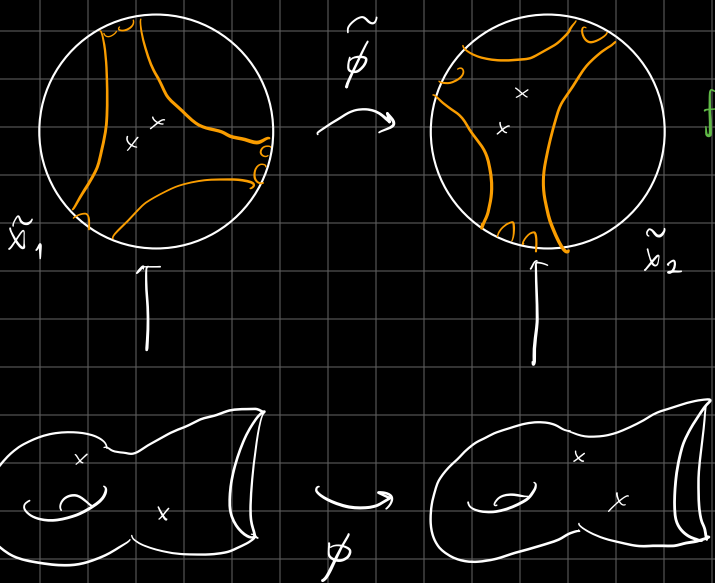
$$\begin{tikzcd} & X_1 \arrow[d, "\phi"] \\ S \arrow[ur, "f_1"] \arrow[r, "f_2"'] & X_2 \end{tikzcd}$$

comm. up to homotopy

$(X_1, f_1) \sim (X_2, f_2)$ iff $\exists \phi : X_1 \to X_1$ conformal such that $f_1 \sim f_2$ homotopic.

* orientation-preserving $\Rightarrow$ isometry $\equiv$ homotopy

- $\mathrm{Teich}(S) = \{(X, f)\}/\sim$

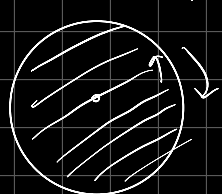


Schottky groups.

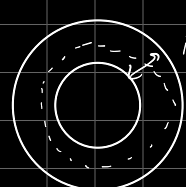
$S \xrightarrow[\text{homeo}]{f} X$ R.S. | can pullback conformal structure to make X into RS

- Base point helpful to serve as origin.

- For orientation-preserving maps, we can assume $\sim$ isotopy.

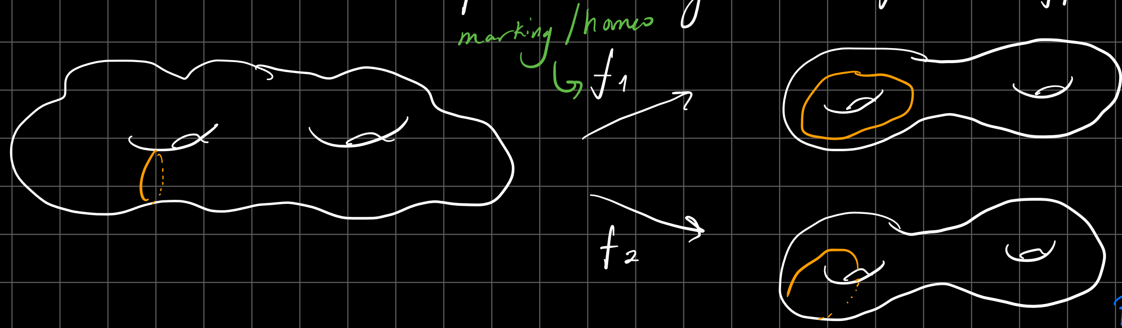


$F(x, t)$
$0 \le t \le 1.$



* can homotope not isotope

- once S is a Riemann surface, we may assume f is a diffeo (C^∞).



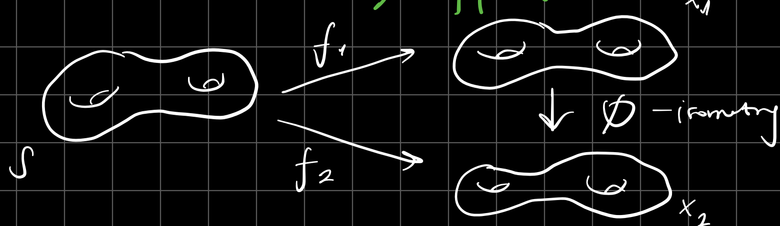
cutting & replacing will not change RS.

But $(X, f_1) \nsim (X, f_2)$

Alternative Description of Teich(S).

- Pulling back the hyperbolic metric → gives an isometry

Hyperbolic metrics on S / $\mathrm{Diffeo}_0^+ S$ ← O.P. diffeos isotopic to id.

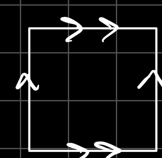


- $f_1^*(d_{hyp}) = (\phi \circ f_1)^*(d_{hyp})$
  $\neq f_2^*(d_{hyp})$

  $f_2^{-1} \circ \phi \circ f_1 \underset{\text{isotopic}}{\sim} id.$

Equivalence classes of hyperbolic metrics

What is Teich($\mathbb{T}^2$)?

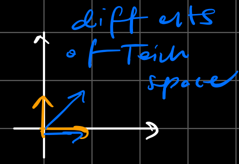


- Flat metric (constant curvature) s.t. { area = 1 }
- Scaling is available in the planar case.

We will show Teich($\mathbb{T}^2$) = $\mathbb{H}$ (or $\mathbb{D}$).

Proof. Λ - lattice in $\mathbb{R}^2$ if it is a discrete subgroup such that $\mathbb{R}^2/\Lambda$ is compact.

$$\mathrm{Teich}(\mathbb{T}^2) \longleftrightarrow (\overbrace{\Lambda, \underbrace{\lambda_1, \lambda_2}_{\text{generators of } \Lambda}}^{\text{marked at}}) / \sim$$



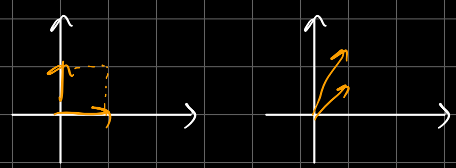
marked lattice.

$(\Lambda, \lambda_1, \lambda_2) \sim (\Lambda', \lambda_1', \lambda_2')$ iff $\exists$ scaling, ordered o.p isometry $(\Lambda, \lambda_1, \lambda_2) \longrightarrow (\Lambda', \lambda_1', \lambda_2')$.

$f(z) = az+b$

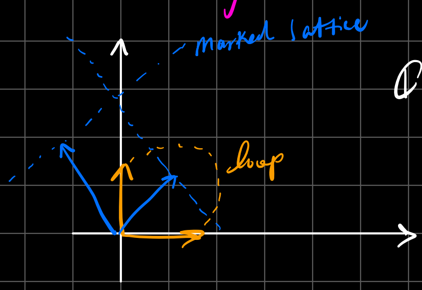
$\boxed{(\longleftarrow)}$ Start with a lattice $(\Lambda, \lambda_1, \lambda_2)$. Define (X, f), where $f: \mathbb{T}^2 \to X = \mathbb{C}/\Lambda$ diffeo.

$$\begin{cases} f(1,0) = \lambda_1 \\ f(0,1) = \lambda_2 . \end{cases}$$



up to scaling, flat metric are same

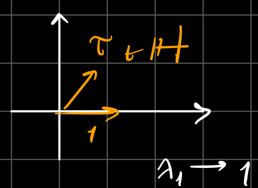
Look at the flat metric on $\mathbb{T}^2$, area 1.



$\mathbb{C}$ $\qquad \pi_1(\mathbb{T}^2) \circlearrowleft \mathbb{C}$

$\rightsquigarrow (\Lambda, \lambda_1, \lambda_2)$

$\boxed{(\longrightarrow)}$ OTOH, take $(\Lambda, \lambda_1, \lambda_2)/\sim$
$\updownarrow$
$\tau \in \mathbb{H}$



$\square$

Think. $\mathrm{Teich}(S_3)$?

Lecture 4 – Thursday, February 20, 2025

- Quasi-conformality

$X \to Y$ closed

Lemma. f a homeomorphism is 1-quasiconformal ($K=1$) if and only if f is conformal.

Proof. ($\Leftarrow$) f conf. means inf circles $\to$ inf circles, which implies df takes circles to circles.

So $K_f = 1$.

($\Rightarrow$) $K_f = 1 \Rightarrow \mu_f = 0$

$$\Rightarrow \frac{df}{d\bar{z}} = 0 \qquad \text{CR equations.}$$

This implies that f is conformal outside of finitely many points.

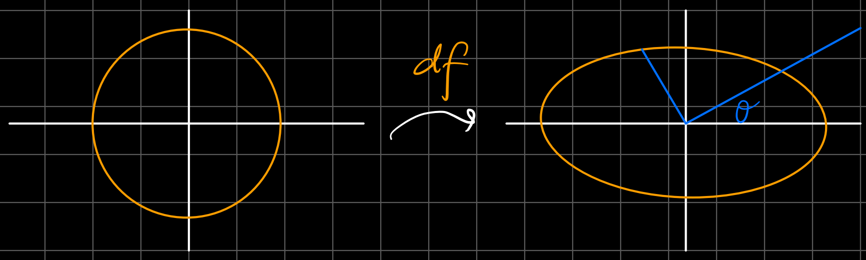
Points are removable for conformal maps.

$\boxtimes$

Note $K=1$-qc $\equiv$ conformal

Properties of Q.C. Maps.

If $f: X \to Y$ homeo K-qc $\Rightarrow f^{-1}$ is K-qc.



major to minor.

Fact. If L - linear map from $\mathbb{R}^2 \to \mathbb{R}^2$ takes a circle to an ellipse with $\frac{\text{major axis}}{\text{minor axis}} = K$.

$\Rightarrow$ L^{-1} takes circles to ellipses with same K.

Ex. $L\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}2x\\3y\end{pmatrix}$, $L^{-1}\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}\frac{x}{2}\\[2pt] \frac{y}{3}\end{pmatrix}$

$K = \frac{3}{2}$

$$\frac{maj}{min} = \frac{\frac{1}{2}}{\frac{1}{3}} = \frac{3}{2}.$$

• If $\underset{K_f \text{ qc}}{X \xrightarrow{f} Y} \overset{K_g \text{ qc.}}{\xrightarrow{g}} Z$ then $g \circ f$ qc

$$K_{g\circ f} \leq K_f \cdot K_g.$$

Exercise. Determine the case of equality.

Idea. If both stretch, then they stretch in the same direction.

Teichmüller's Extremal Problem. $f : X \to Y$ homeo btwn Riemann surfaces.

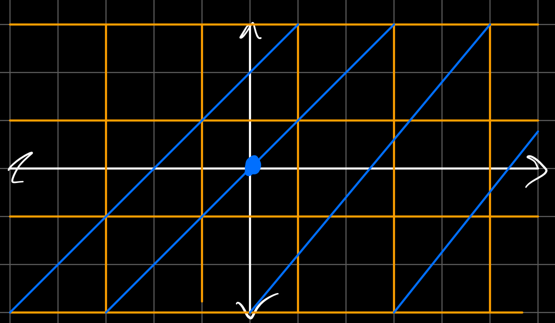
We want to find an h homotopic to f such that h, h^{-1} differentiable at all but finitely many pts.

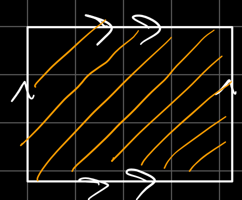
$F(x,t)$ continuous.
$F : X \times [0,1] \to Y$
$(x, 0) \longmapsto f$
$(x, 1) \longmapsto h$

K_h - smallest.
(least amount of stretching)

Teichmüller's Existence & Uniqueness Theorems. (goal)

Measured Foliations on $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$

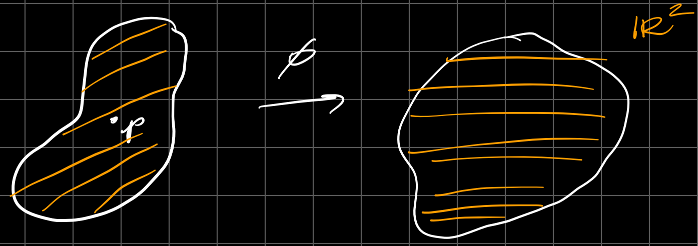




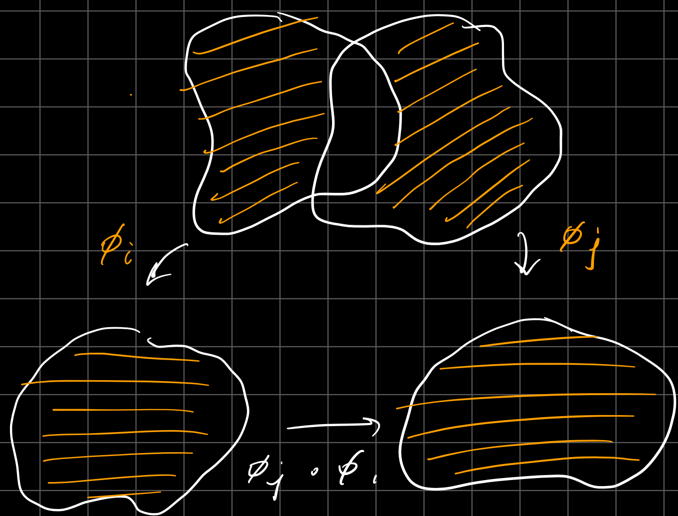
Defn. Let X be a Riemann surface. A FOLIATION of X is a partition

$$X = \bigsqcup F_\alpha \quad \leftarrow \text{LEAVES of } \mathcal{F}$$

such that $\forall p \in X \ \exists (U, \phi)$ chart



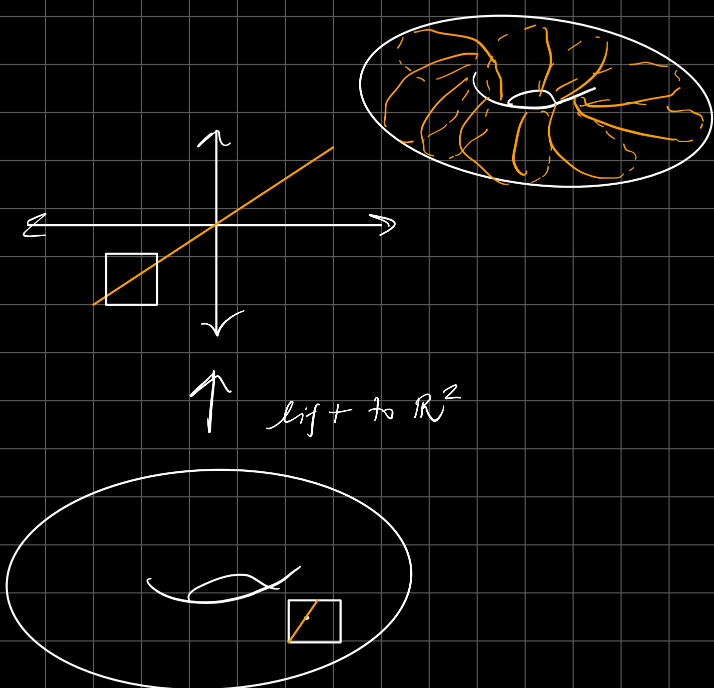
$\phi(F_x \cap U) = \{(x,y) : y\text{-const.}\}$



$\phi_j \circ \phi_i^{-1}(x,y) =$
$(f(x,y), g(y))$.

$F - C^k$ if $\phi_j \circ \phi_i^{-1} - C^k$.

If the slope is rational, then the curve will close. SCC on the T^2.



closes on itself
isn't dense

§11.2.

Ex.

$$\begin{array}{c} \mathbb{R}^2 \\ \downarrow \\ \mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2 \end{array}$$

$A = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$ O.P map $\det = 1$.

induces a map on $\mathbb{T}^2$: proj map.

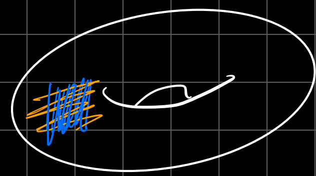
$\phi_A : \mathbb{T}^2 \to \mathbb{T}^2$

$\det(A - \lambda I) = 0 \Rightarrow \lambda = \dfrac{3 \pm \sqrt{5}}{2}$ eigenvalue.

$\lambda_1 \cdot \lambda_2 = 1$

linearly dependent eigenvalues.

eigenspaces are lines.



transverse measured foliation.

$(\phi_A)_* |dv_{e_1}| = \lambda |dv_{e_1}|$ contract

$(\phi_A)_* |dv_{e_2}| = \frac{1}{\lambda} |dv_2|$ expand.

Defn. $(k+2)$-pronged saddle

$\mathrm{Im}(z^{k/2}\, dz) = 0.$

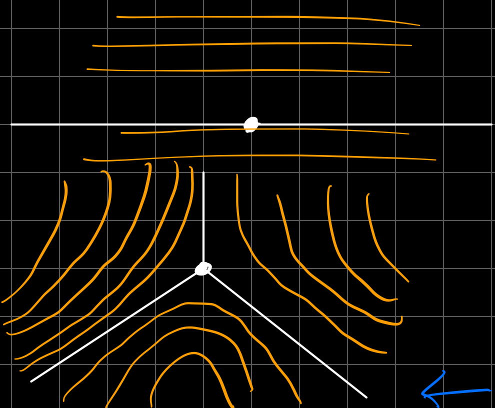
$z = re^{i\theta}$. fix θ. $\mathrm{Im}(r^{k/2} e^{\frac{i\theta k}{2}} e^{i\theta} dr) = 0$

$$\frac{\theta k}{2} + \theta = \pi l, \quad l \in \mathbb{Z}$$

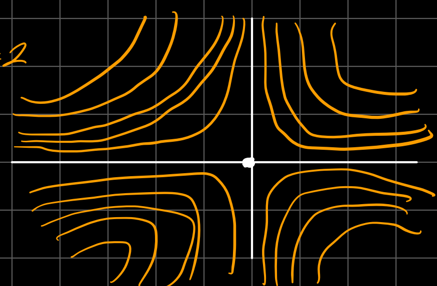
$$\theta = \frac{2\pi l}{k+2}, \quad l = 0, 1, 2, \dots, k+1$$

K = 0

K = 1



K=2

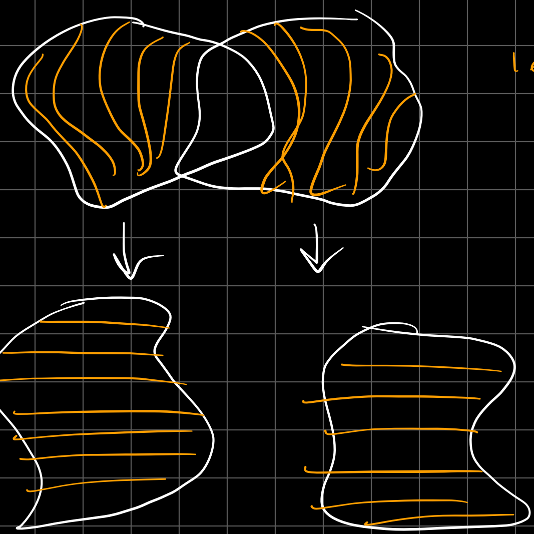


← 3-pronged saddle.

Defn. X = Riemann surface, $\mathcal{F}$ = foliation on X $\quad\Big|\quad X = \bigsqcup_{\alpha} F_\alpha \cup \{\text{finitely many points}\}$

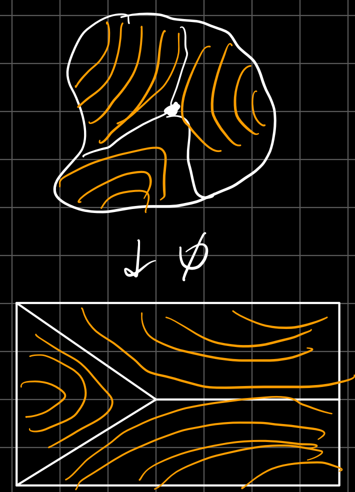
↑ singularity

$\forall p \in X$ not a sing



leaves to horizontal lines

$\forall p$ - sing $\exists (U, \phi)$



Thm. (Euler–Poincaré formula)

$$\chi(X) = \sum_{s = \text{sing}} (2 - k_s)$$

Ex. X = sphere, $k_s \geq 3$.
$\chi(X) > 0$

X = torus
$\chi(X) = 0$

X = genus ≥ 2
$\chi(X) < 0 \implies$ has at least one sing.

No singular foliations on sphere
Only non singular foliations on torus.

Thursday, March 20, 2025

Thm. (Teichmüller's Uniqueness)

Lemma 11.11.

Existence of Teichmüller Maps (indirect proof.)

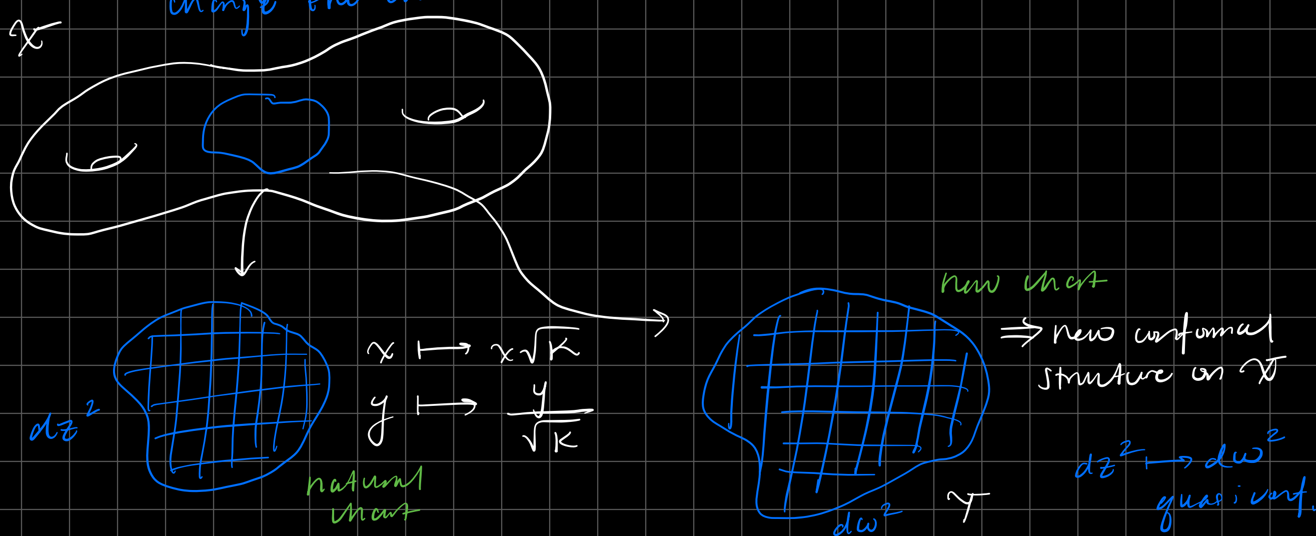
- $\Phi : QD_1(\mathcal{X}) \longrightarrow \mathrm{Teich}(\mathcal{X})$, $q \in QD_1(\mathcal{X})$, $K = \dfrac{1+\|q\|}{1-\|q\|}$

  holomorphic quadratic differentials

  $\|q\| < 1$ — depends on q.

  $\int_{\mathcal{X}} |q|$

  tells you how to change the chart.



$\mathcal{X} \to \mathcal{Y}$ Teichmüller map (stretches in one direction, expands in the other)

$[\mathcal{Y}] \in \mathrm{Teich}(\mathcal{X})$ ← isotopy classes of diffeos.

It suffices to show that Φ is surjective.

1. Φ cont.
2. Φ proper
3. Φ injective. ✓ from Uniqueness thm.

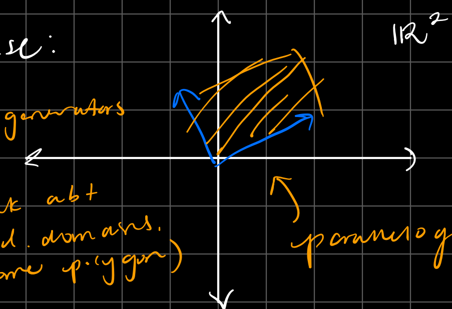
change homotopy class

$\mathbb{H}$

- Brouwer's Invariance of Domain Thm ⇒ Φ homeo and thus onto.
- Topology on Quadratic Differentials $\|q\| = \int_{\mathcal{X}} |q|$. space of isoms
- Topology on $\mathrm{Teich}(\mathcal{X}) = \mathrm{DF}(\pi_1(\mathcal{X}), PSL(2,\mathbb{R})) / PGL(2,\mathbb{R})$

  discrete faithful representations

- T^2 case:

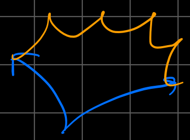


think abt fund. domains. (some polygon)

bounded by geodesics

loop in T^2 lifts to a $\underbrace{\text{translation}}_{\text{isometry}}$ on the plane

- $\mathbb{H}$ case.



Closed to each other in the Hausdorff space.

- $\mathcal{S}$ is proper (preimage of a cpt set is cpt)

Let L be compact in $\text{Teich}(X)$

$= \{[(\underbrace{Y, f)]}_{[Y]}\}$, $f: X \to Y$. Consider $K([Y]) = \inf\{K_h : h \sim f,\ h: X \to Y\}$

q.c dilation.

Claim. $K : \text{Teich}(X) \to [1, +\infty)$ is cont.

Proof. $f: X \to Y$, $\exists\,(K([Y]) + \varepsilon)$ q.c.

K, Y' composition of q.c maps is q.c

$[Y']$ close to $[Y] \Rightarrow \exists h'$ K-q.c $Y \to Y'$ s.t. K-close to 1.

$\exists\, K.\,((K[Y]) + \varepsilon)$ -q.c map. $X \to Y'$

$\varepsilon > 0$ arb $K > 1$ arb close to 1

$\Rightarrow K$ is continuous. $\square$

Back to L compact in $\text{Teich}(X)$. Let M be the max of $K([X])$, $[Y] \in L$.

$\mathcal{S}^{-1}(L)$ is compact? Let $q \in \mathcal{S}^{-1}(L)$. (in $QD_1(X)$) $\underset{\text{Teichmuller map}}{h} : X \to \mathcal{S}(q) \in \text{Teich}(X)$. $\in L$.

$$K_h = \frac{1 + \|q\|}{1 - \|q\|} \leq M.$$

$\|q\| \le \frac{M-1}{M+1}$. Once we show $\mathcal{B}$ is continuous, we have that $\mathcal{B}^{-1}(L)$ is compact.

It remains to show that $\mathcal{B}$ is continuous.

$$\mathcal{B} : QD_1(X) \longrightarrow \mathrm{Teich}(X).$$

We will show that $\mathcal{B} : QD_1(X) \xrightarrow{\mathcal{B}_1} L^\infty(X) \xrightarrow{\mathcal{B}_2} \mathrm{Teich}(X)$.
($\mathcal{B}$ factors/splits)

↑ Measurable Riemann Mapping Theorem

$X \xrightarrow{f} Y$ q.c.

$df_p : \bigcirc \to \bigcirc$

$$df = f_z\, dz + f_{\bar z}\, d\bar z$$
$$= f_z\, dz \left(1 + \frac{f_{\bar z}}{f_z} \frac{d\bar z}{dz}\right)$$

μ_f.

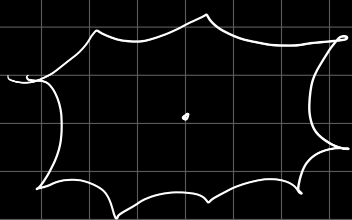
$|\mu_f| = \left|\frac{f_{\bar z}}{f_z}\right|$, Beltrami form

$|\mu_f| \le k < 1$.

L^∞ - norm < 1

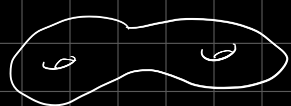
. . .

"Pullback to the universal cover"



$\mu(p(z)) \dfrac{\bar{\phi}'_z}{\phi'_z}$

$p \downarrow$ conf. loc.



X $\qquad \mu$, $|\mu| < 1$

$w = \gamma(z)$

↑ covering transform.

⋮

Thm. (MRM) Given a μ-Beltrami form on $\mathbb{C}$, there exists an $f : \mathbb{C} \to \mathbb{C}$ quasiconformal, $\dfrac{f_{\bar z}}{f_z} \dfrac{d\bar z}{dz} = \mu$.

If $f(0) = 0$, $f(1) = 1$ ($f(\infty) = \infty$), then f is unique.

Moreover, f depends "analytically" on μ.

Goal. Construct $L_1^\infty(X) \xrightarrow{\Phi_2} \mathrm{Teich}(X)$.

Suppose f_μ is the unique solution to $\frac{f_{\bar z}}{f_z} = \mu$.

$\begin{cases} f(0)=0 \\ f(1)=1 \\ (f(\infty)=\infty) \end{cases}$. $\gamma \in \pi_1(X)$ - deck transformation $\circlearrowleft \mathbb{H}$.

$\Rightarrow \boxed{f_\mu(\gamma(z)) = \gamma'(f_\mu(z))}$ Möb of $\mathbb{H}$.

Show: $f_\mu \circ \gamma \circ f_\mu^{-1}$ is conformal in $\mathbb{C}$.

$\frac{(f_\mu)_{\bar z}}{(f_\mu)_z} = \mu$, μ - $\pi_1(X)$ equiv.

Exercise. Show $\frac{\partial}{\partial \bar z}(f_\mu \cdot \gamma \cdot f_\mu^{-1}) = 0$ in $\mathbb{C}$.

— —

$\Gamma \to \Gamma'$ isomorphism
$\wr\wr$
$\pi_1(X)$

$X \rightsquigarrow \pi_1(X)$
$\wr$
Γ' : $\mathbb{H} \circlearrowright$

acts discretely and freely

$Y = \mathbb{H}/\Gamma'$.

$f_\mu : X \to Y$ q.c. descends to quotient bc $(f_\mu \circ \gamma = \gamma' \circ f_\mu)$

$[(Y, f_\mu)] \in \mathrm{Teich}(X)$

continuous (moreover part of MRMT)

$\Phi_2(\mu_1) = [(Y, f_\mu)]$
$L_1^\infty(X)$ $\quad$ $\mathrm{Teich}(X)$

$\Phi_1 : QD_1(X) \to L_1^\infty(X)$
q

X
$\uparrow p$
, $q(p(z)p'(z))^2$

Pull q back to $\mathbb{H}$.

$\pi_1(X)$-equiv. qd on $\mathbb{H}$. $\tilde q(w)\,dw^2 = \tilde q(z)\,dz^2$

$w = \gamma(z)$
$\gamma \in \pi_1(X)$

$\tilde{q}$ - holo q diff on $\mathbb{H}$
$= p^*(q)$.

$\mathcal{B}_1(q) = \|q\| \dfrac{\overline{\tilde{q}(z)}}{|\tilde{q}(z)|}$

Beltrami coeff.
on $\mathbb{H}$, $\pi_1(X)$ equiv.

constant.

Let $w = \gamma(z)$. $\|q\| \dfrac{\overline{\tilde{q}(w)}}{\tilde{q}(w)} \dfrac{\overline{w'}}{w'} \overset{?}{=} \|q\| \dfrac{\overline{\tilde{q}(z)}}{\tilde{q}(z)}$.

$w = \gamma(z)$

$$\text{LHS} = \|q\| \frac{\overline{\tilde{q}(w)}}{|\tilde{q}(w)|} \cdot \frac{\overline{w'}}{w'} \left(\frac{\overline{w'}}{\overline{w'}} \right)$$

$$= \|q\| \frac{\overline{\tilde{q}(w)(w')^2}}{|\tilde{q}(w)(w')^2|}$$

$$= \|q\| \frac{\overline{\tilde{q}(z)}}{|\tilde{q}(z)|} \checkmark.$$

$\mathcal{B}_1(q)$ = Beltrami form on X that is a quotient of $\|q\| \dfrac{\overline{\tilde{q}}}{|\tilde{q}|}$ to X.

By examination, $\mathcal{B}_1$ is cont.

It remains to show that $\mathcal{B} = \mathcal{B}_2 \circ \mathcal{B}_1$.

Thursday, March 27, 2025

- $\dfrac{1+\|q\|}{1-\|q\|} \rightsquigarrow$ stretch factor for q.

Teichmüller Distance.

$$\begin{tikzcd} & X_1 \arrow[dd, dashed] & {[(X_1, f_1)]} \\ S_g \arrow[ur, "f_1"] \arrow[dr, "f_2"'] & & \\ & X_2 & {[(X_2, f_2)]} \end{tikzcd}$$

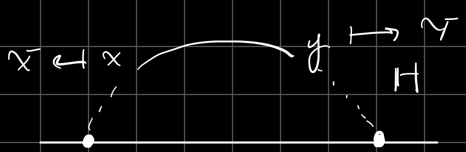
$\exists!\, h \sim f_2 \circ f_1^{-1}$ (h - Teich map)

Defn. TEICHMÜLLER DISTANCE $d_T([(X_1, f_1)], [(X_2, f_2)]) = \frac{1}{2}\ln K_h$.

Thm. $(\mathrm{Teich}(S_g), d_T)$ is a complete (closed balls are compact) metric space.

Thm. $(\mathrm{Teich}(T^2), 2d_T) \underset{\text{isometric}}{\cong} (\mathbb{H}, d_{hyp})$

Pf.



$\dfrac{|dz|}{y}$

$\star$ MCGs $\circlearrowleft$ $\mathrm{Teich}(S)$ by isometries.

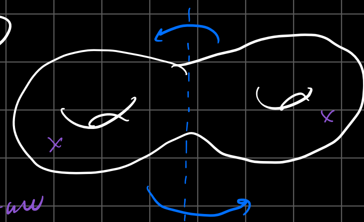
Mapping Class Groups.

- $S_{g,b,n} = S$. $\mathrm{Homeo}^+(S)$ orientation-preserving homeomorphisms of S
  - $\to f|_{\partial S} = \mathrm{id}$
  - $\to f$: marked pts $\mapsto$ marked points (may shuffle)
  - $\to f_1 \sim f_2$ homotopy

Defn. Mapping Class Group $\mathrm{Mod}(S) = \mathrm{Homeo}^+(S, \partial S) / \mathrm{Homeo}_0(S, \partial S)$

Versions. Homotopy $\longleftarrow\!\shortmid$ isotopy
S with differentiable structure, homeos $\longleftarrow\!\shortmid$ diffeos.

Examples.

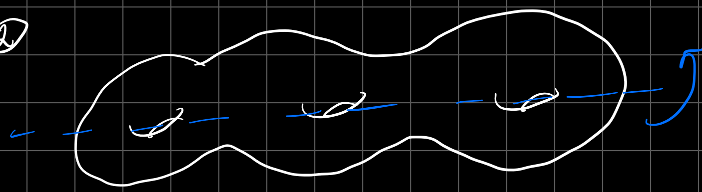
① 

surface fixed mapping of pts

representative of equiv class of mapping class group

order 2.

$\mathrm{Mod}(S_2)$

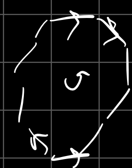
② 

$180° = \pi$, $2g+2$ fixed pts
order 2
hyperelliptic involution.

③ $(4g+2)$ regular polygon

Identify opposite sides.

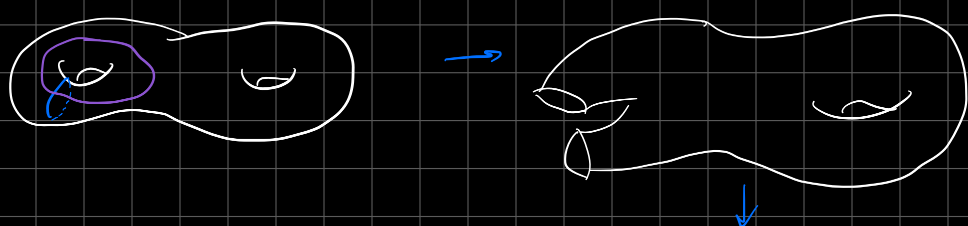
f - rotation by $\dfrac{2\pi}{4g+2}$ angle



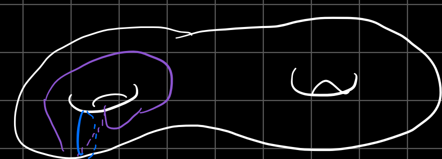
order $(4g+2)$ - elt of $\mathrm{Mod}(S_g)$

Rmk. Rotate by π , hyperelliptic involution.

★ Not all elements of $\mathrm{Mod}(S)$ are of finite order, e.g Dehn twist

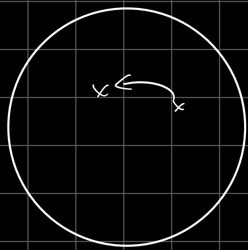
④ 

infinite order element.



Thm. (Alexander's Trick) $\mathrm{Mod}(D^2) = \{pt.\}$.

Pf.



$f \in \mathrm{Homeo}^+(D^2)$

$f|_{\partial D^2} = id.$

$$f(x,t) = \begin{cases} (1-t)\, f\left(\frac{x}{1-t}\right), & 0 \le |x| < 1-t \\ x, & 1-t \le |x| \le 1. \end{cases}$$

$\square$

⑤ $\mathrm{Mod}(D^2 / \{pt\}) = \text{trivial}.$

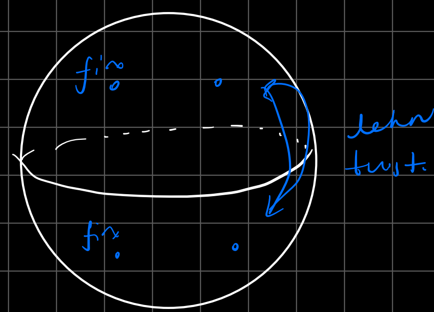
⑥ $\mathrm{Mod}(\overset{= S_0}{S^2}) = \text{trivial}$

⑦ $\mathrm{Mod}(S_{0,\underset{\text{puncture}}{1}}) = \text{trivial}$

⑧ $\mathrm{Mod}(S_{0,2}) = \mathbb{Z}/2\mathbb{Z}$

⑨ $\mathrm{Mod}(S_{0,3}) = S_3$

⑩ $S_{0,4}$



infinite group.

Thursday, April 10, 2025

Back to Fricke's

12.2

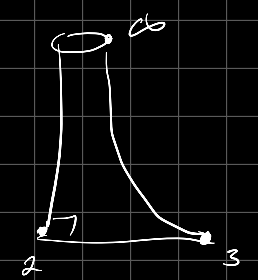
Thm. $\mathrm{Mod}(S_g) \circlearrowleft \mathrm{Teich}(S_g)$ is properly discontinuous.

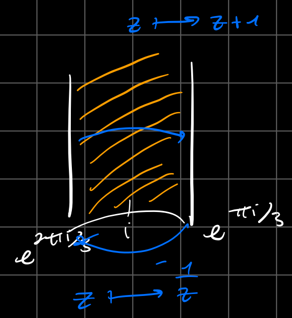
discretely

Cor. The Teichmüller space induces a metric on the moduli space of S_g.

Last time

- $M(T^2)$

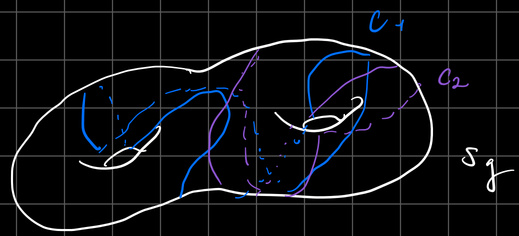


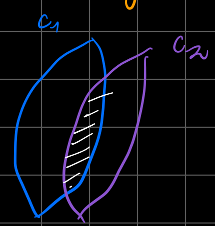


Pf. **Fact.** There exist 2 simple closed curves c_1, c_2 in S_g

→ Minimal position (smallest # of intersection) no bigons

→ Fill S_g



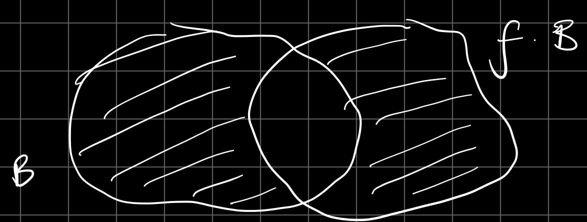


To fill S_g: $S_g \setminus \{c_1 \cup c_2\} = \cup$ of top. disks

To show. $\forall B \subset \mathrm{Teich}(S_g)$ compact

$\#\{[f] \in \mathrm{Mod}(S_g) : f \cdot B \cap B \neq \emptyset\}$ is finite. If $x \in \mathrm{Teich}(S_g)$, $f \cdot x = [(X, \phi \circ f^{-1})]$

$x = [(X, \phi)]$



Let $D = \mathrm{diam}(B)$ in Teichmüller metric. Let $x \in B$, $[f] \in F$.

$x = [(X, \phi)]$

Let $L = \sup\limits_{x \in B} \max\{\ell_x([c_1]), \ell_x([c_2])\}$

geodesic homotopic to c_2 $\sim c_2$

Since $f \cdot B \cap B \neq \emptyset \Rightarrow d_T(x, f\cdot x) \le 2D$,

$x = [(X, \phi)]$, $\quad f \cdot x = [(X, \phi \circ f^{-1})]$

$$\begin{tikzcd} & X \arrow[d, "\exists\, h\text{-}K \text{ qc (Teich)}"] \\ S_g \arrow[ur, "\phi"] \arrow[r, "\phi\circ f^{-1}"'] \arrow[loop left, "f"] & X \end{tikzcd}$$

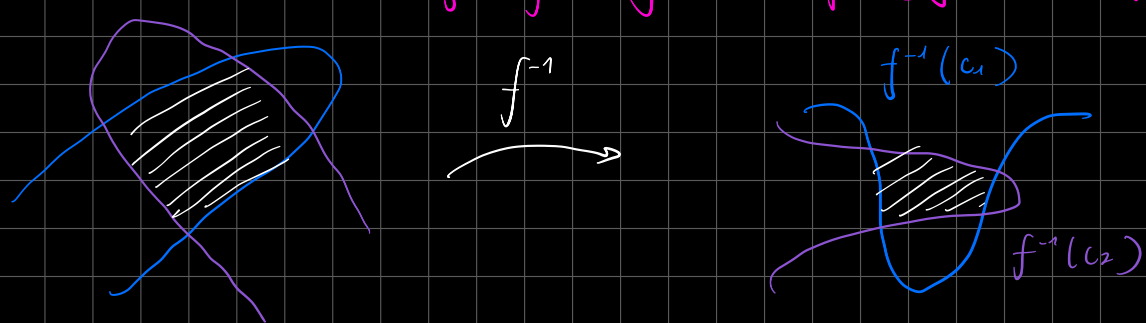
$d(x, f\cdot x) = \frac{1}{2}\ln K \le 2D$

$\Rightarrow K \le e^{4D}$

By Wolpert's lemma,

$$\ell_{fx}(\underbrace{[f^{-1}(c_1)]}_{f\cdot x}), \ \ell_{fx}(\underbrace{[f^{-1}(c_2)]}_{f\cdot x}) \le K \ell_x(c_1), \ell_x(c_2)$$

By 1st lemma, there are finitely many choices for $[f^{-1}(c_1)], [f^{-1}(c_2)]$.



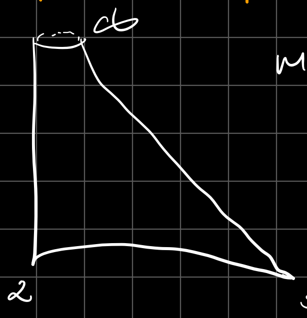
f - homeo $\Rightarrow$ c_1, c_2 - fill S_g

$\Rightarrow f^{-1}(c_1), f^{-1}(c_2)$ fill S_g.

Now use the Alexander trick ($\mathrm{Mod}(D) = \mathrm{id}$) to conclude F is finite.

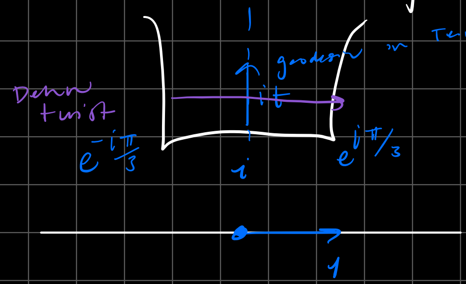
Pigeonhole Principle + Alexander trick $\Rightarrow$ ↯ $\square$

Mumford's Compactness.



$\mathcal{M}(T^2)$ is not compact (has ∞-diameter).

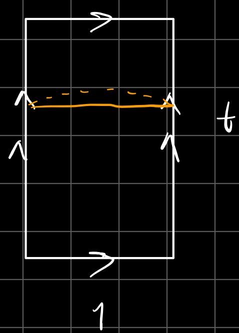
→ Fundamental domain for T^2

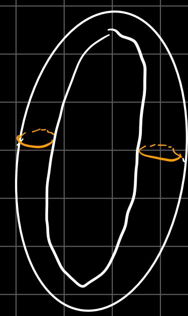


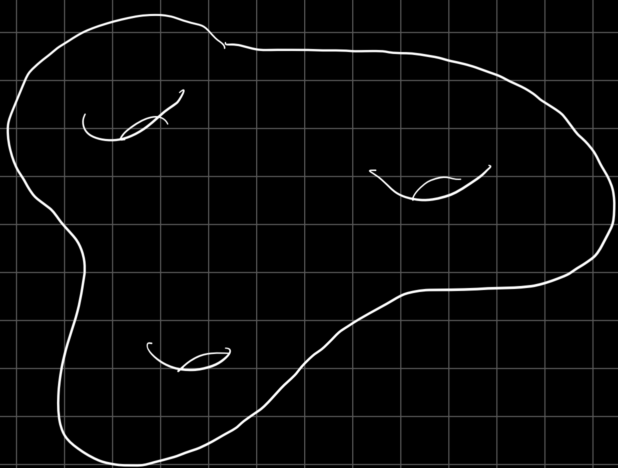
geodesic in Teich space proj into Mod space as geodesic.

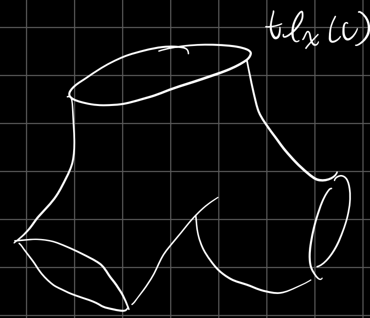
t-large

In Teichmüller space, a Dehn twist sends to ∞, not in Moduli space (need to pinch).









Defn. Let $\varepsilon > 0$. Then $\mathcal{M}_\varepsilon(S_g) = \{x \in \mathcal{M}(S_g) : \ell_x(x) \geq \varepsilon\}$

length of smallest geodesic in x.

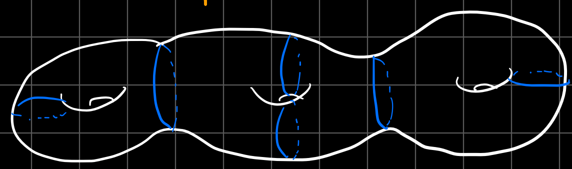
$\mathcal{M}_\varepsilon(S_g)$: ε-thick part of $\mathcal{M}(S_g)$

$\mathcal{M}(S_g) = \bigcup_{\varepsilon > 0} \mathcal{M}_\varepsilon(S_g)$

Thm. (Mumford's compactness) $\forall \varepsilon > 0$, $\mathcal{M}_\varepsilon(S_g)$ is compact in $\mathcal{M}(S_g)$.

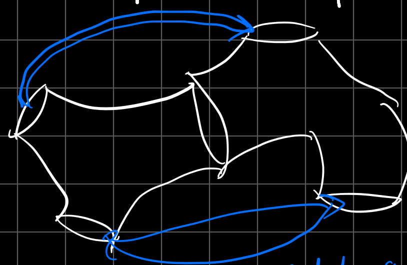
Thursday, April 24, 2025

Pants Decomposition

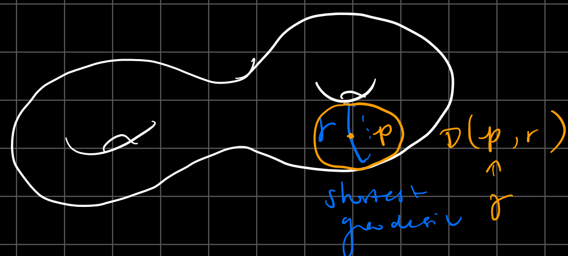


$3g-3+b$ curves in pants decomposition of $S_{g,b}$

- T^2 cannot be decomposed into pairs of pants



- Area of a hyperbolic surface $= 2\pi\chi(X)$ ← Euler characteristic



$r = \tfrac{1}{2} l_X(\gamma)$

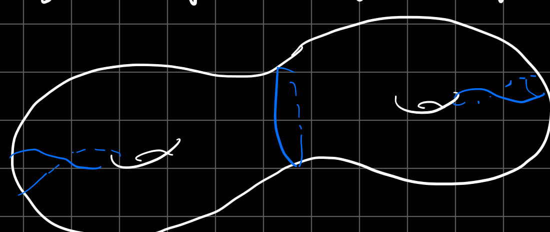
- $\underbrace{\text{Area}(D)}_{\int_0^{2\pi}\int_0^r \sinh(t)\,dt\,d\theta} \leq -2\pi\chi(S)$

Upper bound

$l_X(\gamma) \leq c(\chi(S))$

Induction...

Fact. There exist only finitely many topological types of pants decomposition for S_g.



· $\mathrm{Teich}(S_g)$ ← Riemann surface with markings

(lift / branch covering)

$\downarrow$

$\mathcal{M}(S_g)$

Nielsen – Thurston for the Torus T^2

$\mathrm{Teich}(T^2) = \mathbb{H}^2$

$\mathrm{Mod}(T^2) = SL(2,\mathbb{Z})$

$f \in \mathrm{Mod}(T^2)$ mapping class

$$f \longleftrightarrow A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in SL(2,\mathbb{Z})$$

$\updownarrow$

Preserves angles

Takes circles to circles

Möbius transformation $\gamma = \dfrac{az+b}{cz+d} \in PSL(2,\mathbb{Z})$

(isometry of $\mathbb{H}^2$, i.e. $\gamma \in \mathrm{Isom}^+(\mathbb{H}^2)$)

→ Can extend to the boundary

→ Homeo to closed ball (compactification)

3 options:

① γ has a fixed pt in $\mathbb{H}^2$ PERIODIC

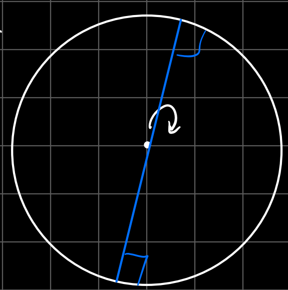
② γ has one fixed pt on $\mathbb{R} \cup \{\infty\}$ REDUCIBLE

③ γ has two distinct fixed pts on $\mathbb{R} \cup \{\infty\}$ ANOSOV (T^2 has no singularities)

(fixed pt $\Rightarrow \dfrac{az+b}{cz+d} = z$

solve quadratic)

① $\mathbb{H}$



$\mathbb{D}$

By conjugating 0-fixed pt, $\gamma = \alpha z$, $|\alpha| = 1$

$\mathrm{Mod}(T^2)$ acts discretely

$\Rightarrow \gamma$ – finite order around fixed pt.

② By conjugation, assume fixed pt is ∞

$\mathbb{H}^2$

$\gamma = \dfrac{az+b}{cz+d}$ $\quad \gamma(\infty) = \infty$

$\Rightarrow c = 0$

$\gamma = az+b$

$az+b = z \Rightarrow az - z = -b \Rightarrow z(a-1) = -b$

$z = -b/(a-1)$

$a \neq 1 \Rightarrow \exists x$ fixed pt. $\Rightarrow \gamma = z + b$.

$\gamma \longleftrightarrow A = \pm \begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix}$

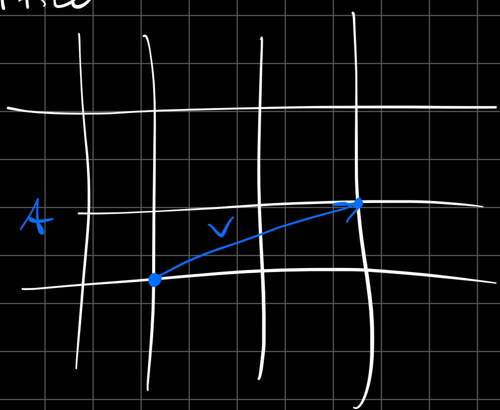
Either $\lambda = 1$ or $\lambda = -1$ as eigenvalue.

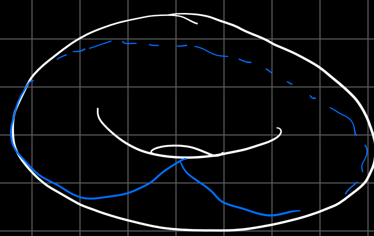
$\lambda = 1 \Rightarrow Av = v$
$\lambda = -1 \Rightarrow Av = -v$,

Let v be the corresponding eigenvector.

$A \in SL(2, \mathbb{Z})$, $c \cdot v$, $c \in \mathbb{Z}$

$\circlearrowleft$ Lattice

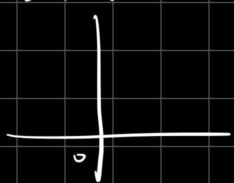




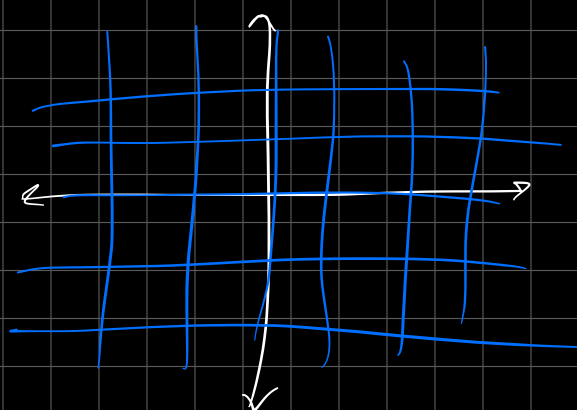
Simple closed curve (unoriented)

$\exists$ c - simple closed curve on T^2 such that $f[c] = [c]$

(3) $\gamma(0) = 0$, $\gamma(\infty) = \infty$
$\gamma(z) = a \cdot z$, $a \overset{\neq 1}{>} 0$



$A = \pm \begin{bmatrix} \sqrt{a} & 0 \\ 0 & \frac{1}{\sqrt{a}} \end{bmatrix}$



preserves traverse foliations (vertical & horizontal)

$A \circlearrowleft$

$(\mathcal{F}^u, \mu^u)$ unstable (horizontal)

$(\mathcal{F}^s, \mu^s)$ stable (vertical)

$$\frac{\sqrt{a}\, z}{\frac{1}{\sqrt{a}}} = a z.$$

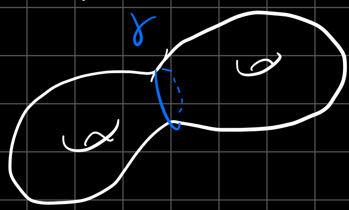
$\lambda = \sqrt{a}$ stretch factor

Thursday, May 1, 2025

- $f \in \mathrm{Mod}(S)$ reducible if $\exists \underbrace{\{c_1, \dots, c_n\}}_{\text{reduction system}}$, $n \in \mathbb{N}$ homotopy classes of sccs such that $i(c_i, c_j) = 0$ & $f(c_i) = c_i$ $\forall i = 1, \dots, n$.

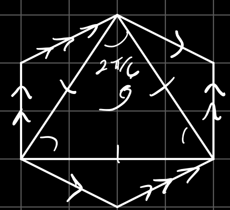
Example.

σ rotation of order 2



T_γ Dehn twist

σ, T_γ reducible
σ also periodic
T_γ not periodic



"Hexagonal" Torus
Periodic (period = 3)
Not reducible.

Defn. $f \in \mathrm{Mod}(S)$ is PSEUDO-ANOSOV if $\exists \phi \sim f$ (p-A homeo) & $\exists (\mathcal{F}^u, \mu^u), (\mathcal{F}^s, \mu^s)$ transverse measured foliations unstable and stable & $\exists \lambda > 1$ stretch factor

$$\phi(\mathcal{F}^u, \mu^u) = (\mathcal{F}^u, \lambda \mu^u)$$
$$\phi(\mathcal{F}^s, \mu^s) = (\mathcal{F}^s, \tfrac{1}{\lambda} \mu^s)$$

unique up to conjugation.

Natural charts, Teichmüller map.

Canonical Form of the NTC.

$f \in \mathrm{Mod}(S_{g,n})$
$\{c_1, \dots, c_m\}$ canonical reduction system (maybe empty)
$\exists R_1, \dots, R_m$ closed annular neighborhoods
$\gamma_1 \in c_1, \dots, \gamma_m \in c_m$
representatives

$R_i \cap R_j = \emptyset$ for $i \neq j$

Let R_j be the closure of a connected component of $S \setminus \bigcup_{i=1}^{m} R_i$, $j = m+1, \dots, p$

embedding $\exists \eta : \mathrm{Mod}(R_i) \to \mathrm{Mod}(S)$
$\forall i = 1, \dots, p$.

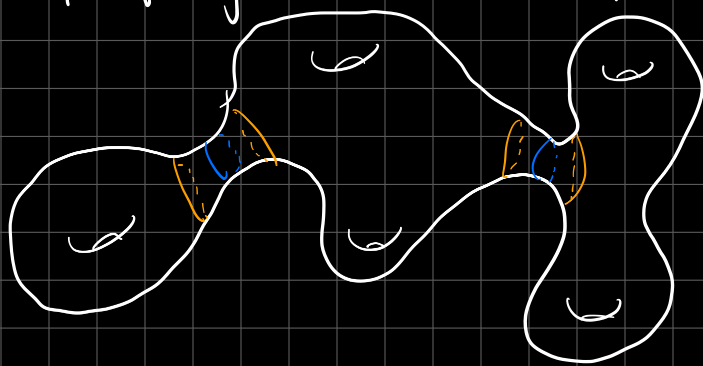
$\exists\, k \in \mathbb{N}.\ f^k \sim \prod_{i=1}^{m} \eta_i(f_i) \prod_{i=m+1}^{p} \eta_i(f_i),\ f_i \in \mathrm{Mod}(R_i)$

$\eta_i(f_i)$ - powers

(+, -, or power of Dehn twist) for $i = 1, \dots, m$

$\eta_i(f_i) = id$ or p-A for $i = m+1, \dots, p$

$m = 2$
$p = 5$



Recall. $\mathrm{Mod}(S) \circlearrowleft \mathrm{Teich}(S)$ by isometries.

Defn. $\mathcal{X}$ = metric space. $F \in \mathrm{Isom}(\mathcal{X})$.
$\tau(F) = \inf_{x \in \mathcal{X}} \{ d(x, F(x)) \}$ TRANSLATION LENGTH of F.

Defn. $\underset{\neq\ id}{F}$ is ELLIPTIC if $\tau(F) = 0$, $\underbrace{\text{realized}}_{\inf = \min}$

F is PARABOLIC if $\tau(F)$ is not realized, inf but not min.

F is HYPERBOLIC if $\tau(F) > 0$, not realized.

We will show. ① parabolic $\Rightarrow$ reducible
② hyperbolic $\Rightarrow$ pseudo-Anosov.

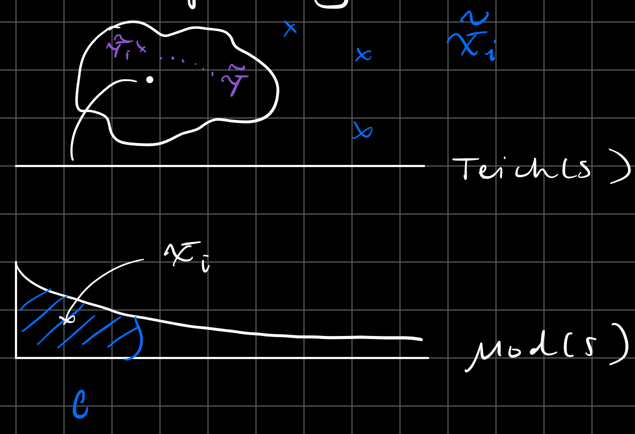
elliptic $\Rightarrow$ periodic (skip)

① $\exists\ \tilde{\mathcal{X}}_i \in \mathrm{Teich}(S = S_{g,n})$. $d_T(x_i, f \cdot x_i)$ converges to $\tau(f)$

Let $\underset{\mathcal{M}(S)}{x_i} = p(\underset{\mathrm{Teich}(S)}{\tilde{x}_i})$ | $p: \mathrm{Teich}(S) \to \mathcal{M}(S)$

Claim. (x_i) leaves every compact subset of $\mathcal{M}(S)$.

Pf. Suppose not. Then there exists C - compact in $\mathcal{M}(S)$ infinitely-many x_i in C.



$\exists\, C' \subset \mathrm{Teich}(S)$ compact.

$\exists\, h_i \in \mathrm{Mod}(S)$

$\tilde{\gamma}_i = h_i \cdot \tilde{x}_i \in C' \subset \mathbb{R}^n$

$\Rightarrow \exists\, i_k : \tilde{\gamma}_{i_k} \to \tilde{\gamma} \in C' \in \mathrm{Teich}(S)$.

$$d_T(\tilde{\gamma}_i,\ h_i f_i h_i^{-1} \tilde{\gamma}_i) = d_T(h_i^{-1}\tilde{\gamma}_i,\ f_i h_i^{-1} \tilde{\gamma}_i)$$

$$= d_T(\tilde{x}_i,\ f_i \tilde{x}_i) \xrightarrow[\text{converges}]{} \tau(f).$$

Subclaim. $\exists\, N$ s.t. $d_T(\tilde{\gamma},\ h_N \tilde{\gamma}\, h_N^{-1}) = \tau(f)$

$$d_T(\tilde{\gamma},\ h_i f h_i^{-1} \tilde{\gamma})$$

Pf. $\tau(f) \le d_T(\tilde{\gamma},\ h_i f h_i^{-1} \tilde{\gamma}) \overset{\Delta\text{-ineq}}{\le} \cancel{d_T(\tilde{\gamma}, \tilde{\gamma}_i)} \to 0 \ +$

$\cancel{d_T(\tilde{\gamma}_i,\ h_i f h_i^{-1})} \to \tau(f) \ +$

$\cancel{d_T(h_i f h_i^{-1}\tilde{\gamma},\ h_i f h_i^{-1} \tilde{\gamma})} \to 0$

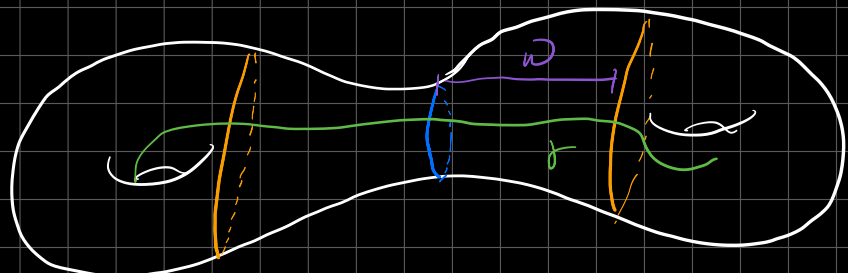
$$\Rightarrow \lim d_T(\tilde{\gamma},\ \underbrace{h_i f h_i^{-1}}_{\in \mathrm{Mod}(S)} \tilde{\gamma}) = \tau(f).$$

Since $\mathrm{Mod}(S)$ acts properly discontinuously (discretely) on $\mathrm{Teich}(S)$, $\exists\, N : d_T(\tilde{\gamma},\ h_N f h_N^{-1} \tilde{\gamma}) = \tau(f)$ $\square$

So $d_T(h_N^{-1}\tilde{Y}, f h_N \tilde{Y}) = d_T(\tilde{X}_N, f X_N)$
$= \tau(f)$.

Realized ↯ $\square$

Lemma. (Collar)



There is $A(\gamma)$-annulus embedded in S and the weight

$$w \geq \sinh^{-1}\left(\frac{1}{\sinh^{-1}(1/2\,\ell(\gamma))}\right).$$

Cor. If β is a simple closed geodesic in S with $i(\gamma,\beta)\neq 0$, then $\ell(\beta)$ - large if $\ell(\gamma)$ - small, quantitatively.

Cor. There exists a $\delta > 0$ such that if $\ell(\gamma), \ell(\beta) \leq \delta$, then γ, β - disjoint.

Claim. + **Mumford's comp.** $\Rightarrow \ell(X_i) \longrightarrow 0$
length of short geodesic in X_i

① Let M be so large that

- $d_T(X_M, f\cdot X_M) < \dfrac{\tau(f)+1}{3g-3+n}$ length of isotopy classes of disjoint simple closed curves.

- $\ell(X_M) < \delta/K$, where $\delta > 0$ is constant from the 2nd cor. to the collar lemma.

Let c_0 be a simple closed curve such that

$$\ell_{X_M}(c_0) = \ell(X_M),$$

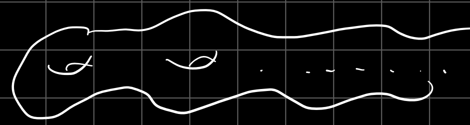
i.e. c_0 cont. a s.c.g., shortest in X_M. Look at $c_i = f^{-1}(c_0)$,

$i = 1, \dots, 3g-3+n.$

$\vdots$

Towards a NTC for Infinite Type Surfaces — Megha Bhat

- Paper by Bestvina, Fanoni, Tao
- S = inf. type surfaces
  $\pi_1(S)$ has ∞ generators
  Teich(S) — bad.



Thm. If $f \in ET(S) \subseteq Mod(S)$, then either
extratame

① f is periodic

② f is a translation (as an isometry of $\mathbb{H}^2/\cdot$)

③ f is "reducible"

- S finite type, $f \in Mod(S)$
  $\forall \alpha, \beta$ simple closed curves

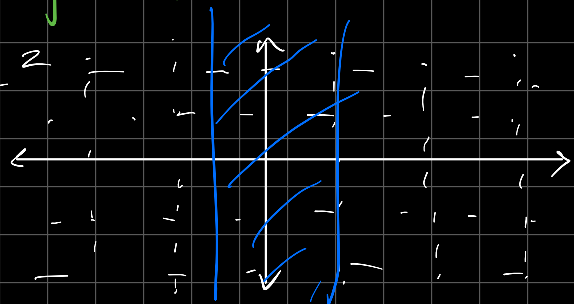
① $i(f^n \alpha, \beta) \to \infty$ (p-A)
② $i(f^n \alpha, \beta)$ bdd $\forall n$ (periodic)

else, reducible.

Defn. $f \in Mod(S)$ is TAME if $\forall K \subseteq S$ finite type, α SCC
$f^n \alpha \cap K$ is one of finitely many isotopy classes.

f is EXTRA TAME if it is tame + $\forall \alpha$, the limit set of α is finite $\{f^n(\alpha)\}$ - limits of subseq.

Examples. $S = \mathbb{R}^2 \setminus \mathbb{Z}^2$



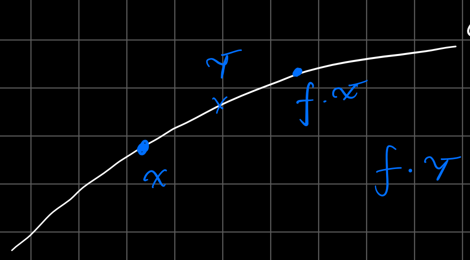
Tame but not extra tame.

Thursday, May 8, 2025

Hyperbolic case.

- $\mathrm{Mod}(S) \circlearrowright \mathrm{Teich}(S)$ by isometries, $f \in \mathrm{Mod}(S)$
  $\tau(f) = \inf d_T(\tilde{x}, f\cdot\tilde{x})$, $x \in \mathrm{Teich}(S)$

Case: Let $\tilde{x} \in \mathrm{Teich}(S)$, $0 < d_T(x, f\cdot x) = \tau(f)$.



$\exists!\ \gamma$ - geodesic in $\mathrm{Teich}(S)$ cont. $x, f\cdot x$.

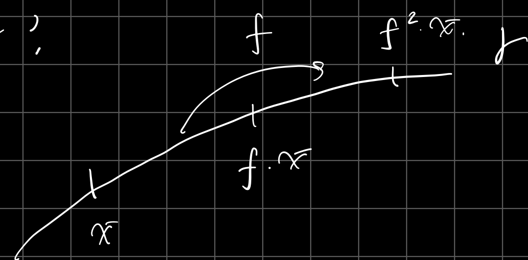
$\forall Y$ in geodesic arc connecting $x, f\cdot x$, $f\cdot Y \in \gamma$ (to show)

$$\tau(f) \le d_T(Y, f\cdot x) \le d_T(Y, f\cdot x) + d(f\cdot x, f\cdot Y)$$
$$= d_T(Y, f\cdot x) + d_T(x, Y)$$
$$= d_T(x, f\cdot x) = \tau(f)$$

$\Rightarrow$ First "$\le$" is "$=$", i.e. $f\cdot x \in \gamma'$ geodesic connection $Y, f\cdot Y$.

$\Rightarrow$ geodesic arc conn. Y & $f\cdot x \subset \gamma, \gamma'$

Uniqueness of geo $\Rightarrow \gamma = \gamma'$.



To show: $f \sim \phi$ is pA.

$\tilde{x} = (x, \psi)$

$$f \circlearrowright S \quad \begin{tikzcd} S \arrow[r, "\psi"] \arrow[dr, "\psi\cdot f^{-1}"'] & x \\ & x' \arrow[u, "\phi"'] \end{tikzcd}$$

$\phi \sim \psi \circ f \circ \psi^{-1}$

Teichmuller map (unique)

Recall. ϕ has 2 quadratic differentials q, q'
initial, terminal

$$\Xi \xrightarrow{\phi} \Xi$$

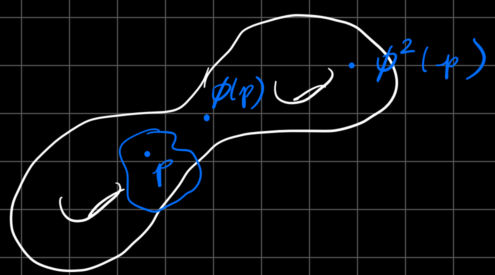
Lemma. If K_ϕ a stretch factor for ϕ, then ϕ^2 is a Teich map whose stretch factor is $(K_\phi)^2$.

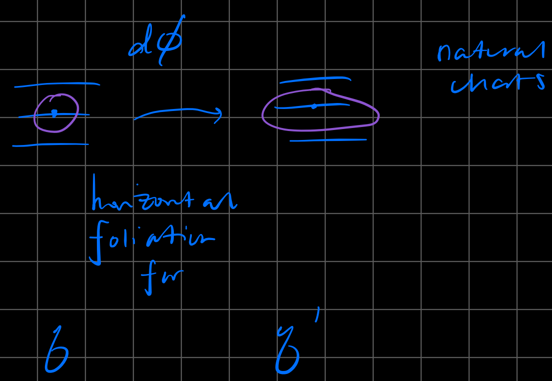
Pf. $K_{\phi^2} \le (K_\phi)^2$

$$d_T(x, f^2 \cdot x) \le \tfrac{1}{2} \ln (K_\phi)^2 = \ln K_\phi$$

$$d_T(x, f^2 \cdot x) = d(x, f \cdot x) + d_T(f \cdot x, f^2 \cdot x) = \tfrac{1}{2} \ln K_\phi + \tfrac{1}{2} \ln K_\phi = \ln K_\phi .$$

So, ϕ^2 – Teich map with stretch factor $(K_\phi)^2$. ☑





Because $K_{\phi^2} = (K_\phi)^2 \Rightarrow$ horizontal foliation for q'
— " — q

$q' = c \cdot q$. norms $= 1$ so $q' = q$ precisely.

$\Rightarrow \phi$ preserves horizontal foliation for $q = q'$ $\mathcal{F}^u$.

$\Rightarrow -q = -q'$ also preserved (vertical foliation) $\mathcal{F}^s$

$\boxtimes$